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STUDY OF THE ROY EQUATIONS
(I). Analyticity, crossing and threshold behaviour

D ATKINSON and T.P POOL
Institute for Theoretical Physics, Postbox 800 wsn 4, Groningen, The Netherlands

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Abstract We study critically the pion dispersion relations of Roy and show that they lead in general to incorrect threshold behaviour for the higher partial-wave amplitudes We are able to modify the equations, in order to cure this disease, but at the expense of a reduced domain of convergence The new equations of Mahoux, Roy and Wanders are found to be free from the difficulty, and we are able to cast them into a form reminiscent of the Cini–Fubini representation

1. Introduction

Roy [1] has introduced twice-subtracted, fixed-t pion dispersion relations, in which the subtraction functions are evaluated by use of st crossing symmetry. If one expands the absorptive parts in partial waves, and then projects the dispersion relations onto Legendre polynomials, one can construe the Roy equations as expressions for the real parts of the partial-wave amplitudes in terms of their imaginary parts, and the two S-wave scattering lengths The equations have been so used by various authors [2–4], who employ unitarity as a constraint, and who make a number of numerical predictions. In particular, the authors of ref. [3] have found only a subset of the solutions claimed by those of ref [2].

The most complete numerical work appears to be that of Basdevant et al. [2], who find that some predictions can be made, although the range of possible solutions of the Roy equations, combined with unitarity, is rather large. This conclusion is to be expected, since the Roy equations are less restrictive than the Mandelstam equations with full crossing symmetry, and it has been demonstrated [5] that there exists a very large infinity of solutions of the latter equations. Nevertheless, the numerical extent of the non-uniqueness has not been explored hitherto in a satisfactory way*, and it may be that the Roy equations provide a practical way to do this in a partial manner

Previous authors, including Roy himself [1–4,7], use Bose symmetry to halve

* See however the preliminary work [6].

the physical interval, $-1 \leqslant \cos\theta \leqslant 1$, in the partial-wave projection. This procedure is acceptable only for an amplitude that has this symmetry, but an approximate or iterative use of the Roy equations does not guarantee this at intermediate steps. and in fact the threshold behaviour of waves with $l \geqslant 3$ is not correctly reproduced in such an iteration. Since the higher waves are coupled to the lower ones, this defect may be expected to affect also the S, P and D waves. It is misleading to suggest, as some authors do, that the S and P waves drop out of the equations identically, since they are dependent upon the higher absorptive parts through the so-called driving terms.

One possible remedy is to employ the whole projection interval $-1 \leqslant \cos\theta \leqslant 1$, although this reduces considerably the domain of validity of the equations A more attractive alternative lies in the use of the new equations of Mahoux et al. [8], in which the Wanders symmetric variables [9] guarantee full crossing symmetry for the amplitudes at each iterative step, so that use of the half-interval is justified Hence no subsidiary conditions are needed to ensure full crossing symmetry, although such conditions are needed for the original Roy equations However, there is a supplementary requirement which takes the place of these subsidiary conditions, namely that the amplitudes be independent of the particular Mahoux–Roy–Wanders equation that one uses. (There is a whole family of equations parametrized by a constant, x_0, as we shall see in sect 4.)

In this introductory paper we examine some of the properties of the Roy equations In future work, we hope to establish the existence of fixed points for these equations, when they are combined with unitarity, and we propose also to investigate the system numerically, in particular to check the conclusions of Basdevant et al. for equations that do not suffer from the threshold disease.

The present paper is arranged as follows in sect. 2 we display the original Roy equations as a mapping, both with the half and with the whole interval, and in sect 3 we show that the correct threshold behaviour is reproduced in the latter, but not in the former case. In sect 4 we give the new equations of Mahoux et al which we cast into the form of a Cini–Fubini representation [10]. We show that the threshold behaviour of the partial waves is correct in this case.

2. Roy equation

Roy evaluated the subtraction function in a twice-subtracted, fixed-t dispersion relation by using st crossing symmetry. The result is

$$F(s,t) = \tfrac{1}{4} g_1(s,t)a + \int_4^\infty ds' [g_2(s,t,s')A(s',0) + g_3(s,t,s')A(s',t)] \ , \qquad (2.1)$$

where F and A are respectively the pion amplitude and its s-channel absorptive part, both written as three-component column vectors, corresponding to the isospin states $I = 0, 1, 2$. Here

$$a = \begin{bmatrix} a_0 \\ 0 \\ a_2 \end{bmatrix}, \tag{2.2}$$

where a_0 and a_2 are respectively the $I = 0$ and $I = 2$ S-wave scattering lengths, and

$$g_1(s, t) = s(1 - C_{su}) + t(C_{st} - C_{su}) + 4C_{su} , \tag{2.3a}$$

$$g_2(s, t, s') = C_{st}\left(\frac{1+C_{tu}}{2} + \frac{2s+t-4}{t-4}\,\frac{1-C_{tu}}{2}\right)\frac{1}{\pi s'^2}$$

$$\times \left[\frac{t^2}{s'-t} + \frac{(4-t)^2 C_{su}}{s'-4+t} - \frac{4t+4(4-t)C_{su}}{s'-4}\right], \tag{2.3b}$$

$$g_3(s, t, s') = \frac{1}{\pi s'^2}\left\{\frac{s^2}{s'-s} + C_{su}\frac{u^2}{s'-u} - \frac{(4-t)^2}{s'-4+t}\left[\frac{C_{su}+1}{2} + \frac{2s+t-4}{t-4}\,\frac{C_{su}-1}{2}\right]\right\} \tag{2.3c}$$

The isospin crossing matrices are

$$C_{tu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{st} = \frac{1}{6}\begin{bmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{bmatrix}, \quad C_{su} = \frac{1}{6}\begin{bmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{bmatrix}. \tag{2.4}$$

We may write a partial-wave series for the absorptive part

$$A(s', t) = \sum_{l'=0}^{\infty} (2l'+1)\,\mathrm{Im} f_{l'}(s') P_{l'}\left(1 + \frac{2t}{s'-4}\right), \tag{2.5}$$

and this is convergent for all $s' \in [4, \infty)$ if $t \in (-28, 4)$ Note that our F and A differ from Roy's by a factor of 4, and that we use the kinematic-singularity-free partial-wave amplitude, which is related to Roy's $a_l(s)$ by

$$f_l(s) = \left(\frac{s}{s-4}\right)^{\frac{1}{2}} a_l(s) \tag{2.6}$$

Roy's next step was to project $F(s, t)$, defined by eq (2.1), onto Legendre polynomials He used tu crossing symmetry to reduce the integration range from $-1 \leqslant z_s \leqslant 1$ to $0 \leqslant z_s \leqslant 1$, where z_s is the s-channel scattering cosine

$$z_s = 1 + \frac{2t}{s-4} \tag{2 7}$$

We shall write

$$f_l(s) = \alpha[\mathrm{Im}\, f, l, s] \overset{\mathrm{def}}{=} \tfrac{1}{2}[1+(-1)^l C_{tu}] \int_0^1 \mathrm{d}z_s P_l(z_s) F(s, t(z_s, s)) \tag{2.8}$$

This expression gives an equation for $f_l(s)$, in terms of the $\mathrm{Im}\, f_{l'}(s')$, that is valid for $-4 \leqslant s \leqslant 60$ Although (2 8) is identically satisfied by a fully crossing-symmetric pion amplitude, it is important to realize that, if we combine (2.8) with unitarity, in order to make a non-linear equation for $\mathrm{Im}\, f_{l'}(s')$, a solution is not guaranteed to be crossing symmetric between the three channels. Although *su* crossing was used explicitly to write the fixed-t dispersion relation, *st* crossing was employed to evaluate the subtraction function, and *tu* crossing was invoked in order to halve the z_s integral range, nevertheless an amplitude constructed from the partial-waves (2.8), let us call it

$$F^\alpha(s, t) = \sum_{l=0}^{\infty} (2l+1) P_l(z_s) \alpha[\mathrm{Im}\, f, l, s] \ , \tag{2 9}$$

would only satisfy *tu* crossing automatically. To ensure full crossing, one would have to impose *st* or *su* symmetry as a subsidiary condition

$$F^\alpha(s, t) = C_{st} F^\alpha(t, s) \tag{2.10a}$$

or

$$F^\alpha(s, t) = C_{su} F^\alpha(u, t) \tag{2.10b}$$

The question as to which crossing conditions are guaranteed by the form of the equations is not merely a technical matter, which we have to render explicit in order to apply a fixed-point theorem, it is also important for a numerical approximation scheme. In fact we shall suggest that the amplitude (2.9), based on eq (2.8), is unsatisfactory, in that the threshold behaviour of the higher partial waves is not guaranteed. We shall then be led to consider alternative systems of equations.

Let us first consider the analyticity properties of $\boldsymbol{F}(s, t)$, as defined by eq (2.1) At first sight it looks as if $g_2(s, t, s')$ has a pole at $t = 4$, but a closer examination shows that this is not so. The function $\boldsymbol{F}(s, t)$ has cuts $4 \leqslant s < \infty$, $4 \leqslant u < \infty$, $4 \leqslant t < \infty$, as expected, but it also has in general an unwanted cut $-\infty < t \leqslant 0$ that arises from the denominator $s' - 4 + t$ in eqs. (2.3b) and (2.3c). Strictly speaking, we cannot infer the t-plane analyticity of $\boldsymbol{F}$ outside the domain of convergence of the series (2.5) for $\boldsymbol{A}$ The important point here is that in general there would be a spurious branch-point at $t = 0$, which is within the range of applicability of (2.5). The terms involving the denominator $s' - 4 + t$ are

$$\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\left[\frac{C_{su}+1}{2}+\frac{2s+t-4}{t-4}\frac{C_{su}-1}{2}\right][C_{tu}A(s',0)-A(s',t)]\frac{(4-t)^2}{s'-4+t}, \qquad (2.11)$$

and so the discontinuity across the unwanted cut is

$$i\left[C_{su}+1+\frac{2s+t-4}{t-4}(C_{su}-1)\right][C_{tu}A(4-t,0)-A(4-t,t)]\,. \qquad (2.12)$$

Now suppose that the absorptive parts in (2.5) satisfy

$$\mathrm{Im}\, f_l^I(s)=0 \quad \text{for} \quad I+l \text{ odd}\,, \qquad (2.13)$$

which follows of course from s-channel Bose symmetry This implies

$$A(x,y)=C_{tu}\,A(x,4-x-y)\,, \qquad (2.14)$$

which means that the second factor in (2.12) vanishes, and hence that there is after all no cut $-\infty < t \leqslant 0$ We shall therefore in future always ensure that A satisfies the Bose condition, whether we are considering a fixed-point theorem or a numerical calculation

In the next section, we shall show that eq (2.8) is unsatisfactory for the partial-wave amplitudes, since for $l \geqslant 3$ the threshold behaviour $(s-4)^l$ is not in general produced by the mapping α of (2 8), although we insert the expected behaviour

$$\mathrm{Im}\, f_{l'}(s') \underset{s'\to 4+}{\sim} (s'-4)^{2l'+\frac{1}{2}}\,, \qquad (2.15)$$

into eq (2 5). The cause of the trouble is the use of the half-range for the partial-wave projection. If we use the following mapping instead of (2.8)

$$f_l(s)=\beta[\mathrm{Im}\, f,l,s] \overset{\text{def}}{=} \tfrac{1}{2}\int_{-1}^{1} dz_s\, P_l(z_s)\, \boldsymbol{F}(s,t(z_s,s))\,, \qquad (2.16)$$

then the correct threshold behaviour is guaranteed for all partial waves, as we also show in sect 3. A disadvantage of the mapping β is that (2.16), with $\boldsymbol{F}$ defined as in (2.1) and A as in (2 5), is valid only in the smaller range $0 \leqslant s \leqslant 32$. Indeed, Roy's reason for using the half-interval was precisely to extend the validity of the partial-wave expansion to $s = 60$. However, in view of the threshold difficulties, it must remain doubtful whether this was a good plan.

It is clear that, if we construct an amplitude, $\boldsymbol{F}^\beta(s,t)$, by means of a partial-wave series like (2 9), but with β in place of α, then

$$\boldsymbol{F}^\beta(s,t)=\boldsymbol{F}(s,t)\,, \qquad (2.17)$$

and so this function is automatically su crossing-symmetric, as one can check from eqs (2.1)–(2.3). However, tu crossing (i.e. Bose symmetry) is not automatic, and it would have to be imposed as a subsidiary condition

$$F^{\beta}(s,t) = C_{tu} F^{\beta}(s,u) . \tag{2 18}$$

Notice that, if we impose Bose symmetry on the input partial-wave absorptive parts (2 13), then the output absorptive parts also are tu symmetric, since

$$\mathrm{Im}\,\beta[\mathrm{Im}\,f, l, s] = \mathrm{Im}\,f_l(s) , \tag{2.19}$$

and so the subsidiary condition is an explicit constraint on the real parts (although this implies that the imaginary parts are indirectly constrained, since the real part of F is defined as a function of the scattering lengths, and the absorptive parts, through eq (2 1)) It is also true for the mapping (2 8) that

$$\mathrm{Im}\,\alpha[\mathrm{Im}\,f, l, s] = \mathrm{Im}\,f_l(s) , \tag{2 20}$$

as we shall now show if s and t are in the s-channel physical region, we see from (2 1) and (2 3) that

$$\mathrm{Im}\,F(s,t) = A(s,t) \tag{2 21}$$

Thus (2 20) follows for $I + l$ even, since

$$\int_0^1 \mathrm{d}x\, P_l(x) P_{l'}(x) = \frac{\delta_{ll'}}{2l+1} , \tag{2.22}$$

for $l + l'$ even, and for $I + l'$ odd there is no contribution from (2 5), because the input absorptive parts satisfy Bose symmetry [eq. (2 13)]. For $I + l$ odd, (2.2) follows trivially from the factor $1 + (-1)^l C_{tu}$ in (2.8).

Unfortunately the β-mapping has an undesirable feature Since the tu crossing (2 18) is not automatic for the dispersive part, an attempt to couple (2.16) with unitarity to define a mapping $\mathrm{Im}\,f_l(s) \to \mathrm{Im}\,f_l(s)$ would not be suitable for a fixed-point proof The reason is that one cannot preserve Bose symmetry, even if one constrains the input absorptive parts by (2.13), and so the spurious branch-cut $-\infty < t \leqslant 0$ of eq (2.11) would not disappear. Thus the partial-wave form of the Roy equation would be vitiated

The solution to the above dilemma is simple we need to replace (2.16) by

$$f_l(s) = \gamma[\mathrm{Im}\,f, l, s] \overset{\text{def}}{=} \tfrac{1}{4}(1 + (-1)^l C_{tu}) \int_{-1}^{1} \mathrm{d}z_s\, P_l(z_s)\, F(s, t(z_s, s)) , \tag{2.23}$$

which amounts to taking the partial waves (2.16) for $I + l$ is even, and dropping those for $I + l$ odd. If we define

$$F^{\gamma}(s,t) = \sum_{l=0}^{\infty} (2l+1) P_l(z_s)\, \gamma[\mathrm{Im}\,f, l, s] , \tag{2.24}$$

then clearly this function has tu symmetry, but we have in general lost su symmetry, just as in the case of the α-mapping. We would then have to impose st or su crossing symmetry as a subsidiary condition ((2.10a or b), with γ in place of α) It is easy to see that

$$F^{\gamma}(s,t) = \tfrac{1}{2}[F^{\beta}(s,t) + C_{tu}F^{\beta}(s,u)] \tag{2.25}$$

The γ-mapping (2.23) suffers from the disadvantage that, like the β-mapping, it is only valid up to $s = 32$, but it is free from the threshold disease of the α-mapping and from the Bose malady of the β-mapping. The eqs. (2.23) and (2.1)–(2.5), combined with unitarity, are suitable for an application of a fixed-point principle, if it is assumed that Im $f_l(s)$ is known for $s \geqslant 32$. At a fixed point, one may think of constraining the scattering lengths, and whatever model one has used for Im $f_l(s)$, $s \geqslant 32$, by means of the subsidiary condition. Here one has the choice of using the st or su constraint (2 10) on F^{γ} itself, or the tu constraint on F^{β}, for even with the γ-mapping one can calculate $F^{\beta} \equiv F$ at a fixed point

It has been remarked that the S- and P-wave absorptive parts cancel out of the tu subsidiary condition [7] (2.18) (which of course has to be applied to $F^{\beta} = F$, and not to F^{α} or F^{γ}) It is also true that they disappear from the st or su subsidiary conditions (2 10), as applied to F^{α} or F^{γ} The cleanest way to see this is to write the Roy equation (2 1) in the approximation in which only S- and P-waves are retained for the absorptive part under the integral, i.e.

$$A(s',t) = \mathrm{Im}\, f_0(s') + 3\left(1 + \frac{2t}{s'-4}\right)\mathrm{Im}\, f_1(s'), \tag{2.26}$$

where Im f_0 has no $I = 1$ component, and Im f_1 has only an $I = 1$ component. The result [8] is

$$F(s,t) = I(t,u) + C_{st}\, I(s,u) + C_{su}\, I(t,s), \tag{2.27}$$

where

$$I(t,u) = \frac{s}{4}a + \frac{s(s-4)}{\pi}\int_4^{\infty} \frac{ds'}{s'(s'-4)(s'-s)}\,\mathrm{Im}\, f_0(s')$$

$$+ \frac{3s}{\pi}(t-u)\int_4^{\infty} \frac{ds'}{s'(s'-4)(s'-s)}\,\mathrm{Im}\, f_1(s') = C_{tu}\, I(u,t) \tag{2.28}$$

This has the form of a twice-subtracted Cini–Fubini approximation [10] to the Mandelstam representation, and one sees that crossing symmetry between the three channels is exact. In this approximation, there is no distinction between F^{α}, F^{β} and F^{γ}, and the subsidiary conditions are automatically satisfied. It is only if there are non-vanishing absorptive parts for $l \geqslant 2$ that the subsidiary conditions are not automatic; but since the S- and P-wave contributions enter precisely as in (2.27), and the subsidiary conditions are linear in $\boldsymbol{F}$, it follows that the scattering lengths and the S- and P-wave contributions cancel out of them. However, we stress again that the S- and P-wave amplitudes are indirectly constrained by the subsidiary conditions, since they depend on the absorptive parts of the higher waves ($l' \geqslant 2$ in eq. (2.5)).

3. Threshold behaviour

In this section, we shall examine the threshold behaviour of the partial-wave amplitudes defined by the three mappings α, β and γ of sect 2. We shall show that the correct form, namely

$$f_l(s) \sim (s-4)^l \,, \tag{3.1}$$

$s \to 4+$, is reproduced for the β-mapping (and therefore trivially also for the γ-mapping), but not for the α-mapping. We propose first to study the β-mapping in detail, in order to demonstrate (3.1)

The partial wave of the β-mapping (2.16) may be divided into the following pieces (when integrated over t):

$$\beta[\mathrm{Im}\, f, l, s] = \delta_{l0}\, J_0(s) + \delta_{l1}\, J_1(s) + \frac{2}{s-4}\frac{1}{\pi}\int_4^\infty \mathrm{d}s' \sum_{\kappa=2}^{4} J_{\kappa l}(s,s') \,. \tag{3.2}$$

The first two terms are respectively contributions to the S and P waves, and originate from the subtraction terms. The S-wave terms are

$$J_0^0(s) = a_0 + \frac{s-4}{4}\frac{2a_0 - 5a_2}{3} - \frac{1}{\pi}\int_4^\infty \mathrm{d}s' \sum_{l'=0}^{\infty} \frac{2l'+1}{3s'(s'-4)} \tag{3.3a}$$

$$\times\ [2(s'+s-2)\,\mathrm{Im}\, f_{l'}^0(s') + 3(2s'+s-4)\,\mathrm{Im}\, f_{l'}^1(s) + 5(2s'-s-4)\,\mathrm{Im}\, f_{l'}^2(s')] \,,$$

$$J_0^2(s) = a_2 - \frac{s-4}{4}\frac{2a_0 - 5a_2}{6} - \frac{1}{\pi}\int_4^\infty \mathrm{d}s' \sum_{l'=0}^{\infty} \frac{2l'+1}{6s'(s'-4)} \tag{3.3b}$$

$$\times\ [2(2s'-s-4)\,\mathrm{Im}\, f_{l'}^0(s') - 3(2s'+s-4)\,\mathrm{Im}\, f_{l'}^1(s') + (2s'+5s-4)\,\mathrm{Im}\, f_{l'}^2(s')] \,,$$

with the isospin one term identically zero. The P-wave contribution is non-vanishing only for isospin one, and it has the form

$$J_1^1(s) = \frac{s-4}{72}(2a_0 - 5a_2) - \frac{s-4}{6\pi}\int_4^\infty ds' \sum_{l'=0}^{\infty} \frac{2l'+1}{6s'(s'-4)}$$

$$\times\ [9\,\mathrm{Im}\,f_{l'}^0(s') + 6\,\mathrm{Im}\,f_{l'}^1(s') - 10\,\mathrm{Im}\,f_{l'}^2(s')] \tag{3 3c}$$

These terms ensure the expected S- and P-wave threshold behaviours, and they need not detain us further

The denominator $s' - t$ in (2.3b), and $s' - u$ in (2 3c) yield, after projection onto Legendre polynomials, the expression

$$J_{2l}(s, s') = Q_l\left(1 + \frac{2s'}{s-4}\right)\sum_{l'=0}^{\infty}(2l'+1)\left\{\left[C_{st}\frac{1+C_{tu}}{2} + C_{st}\frac{1-C_{tu}}{2}\left(1 + \frac{2s}{s'-4}\right)\right]\right.$$

$$\left. + (-1)^l P_{l'}\left(-1 - \frac{2s}{s'-4}\right)C_{su}\right\}\ \mathrm{Im}\,f_{l'}(s'), \tag{3 4}$$

which contains the Legendre function of the second kind This immediately gives the required factor (3.1).

The denominator $s' - s$ (2 3c) does not lead to a Q_l function, but the corresponding contribution to the partial-wave amplitude may be written

$$J_{3l}(s, s') = \frac{s(s-4)}{2s'(s'-s)}\,\alpha^l \sum_{l'=l}^{\infty} \frac{2l'+1}{l'+l+1}\, P_{l'-l}^{2l+1,-1}(1-2\alpha)\ \mathrm{Im}\,f_{l'}(s'). \tag{3 5}$$

Here

$$\alpha = \frac{s-4}{s'-4}, \tag{3.6}$$

and $P_l^{\alpha,\beta}$ is the Jacobi polynomial The series in (3 5) starts at the point $l' = l$, because the contributions from the waves $l' < l$ in (2.5) are orthogonal to $P_l(1 + 2t/(s-4))$, over the whole interval $4 - s \leqslant t \leqslant 0$, this is the source of the factor α^l in (3.5), which guarantees the threshold (3.1). It may be noted at this point that the orthogonality is no longer valid in the case of the α-mapping, where only the half-interval is used in the partial-wave projection. It is also true in this case that the J_2 term no longer gives a Legendre function of the second kind, as in (3.4), and this also gives an incorrect threshold behaviour. After some tedious calculations, one may show in fact that the α-mapping yields the threshold $(s-4)^2$ for $l \geqslant 2$, and hence that the threshold is definitely wrong for $l \geqslant 3$

Finally, the remaining terms in (2.3b) and (2.3c), which contain the denominator $s' - 4 + t$, give the contribution

$$J_{4l}(s, s') = \sum_{l'=l+1}^{\infty} (2l'+1) \sum_{m=l+1}^{l'} \frac{2m+1}{l'+m+1} \alpha^m P_{l'-m}^{2m+1,-1}(1-2\alpha)$$

$$\times \left\{ A_{l,m} \left(-1 - \frac{2s'}{s-4}\right) C_{su} - A_{l,m} \left(1 - \frac{2}{\alpha}\right) \left[C_{su} \frac{s'-s}{s'} + \frac{s}{s'} \right] \right\} \mathrm{Im}\, f_{l'}(s') , \tag{3 7}$$

where

$$A_{l,m}(z) = Q_l(z) P_m(z) - P_l(z) Q_m(z) , \tag{3.8}$$

which is a polynomial, the degree being $m - l - 1$ for $m \geqslant l + 1$ Thus the factor α^m, together with the terms involving $A_{l,m}$, result in the expected behaviour (3.1).

4 Mahoux–Roy–Wanders equation

In this section, we shall first outline the method of Mahoux et al. [8], and we shall cast their new equation into an elegant form In the first place, one uses the Roskies amplitudes [11] ·

$$G_0(s, t) = \tfrac{1}{5}\{F_0(s, t) + F_0(t, u) + F_0(u, s)\} ,$$

$$G_1(s, t) = \frac{F_1(s, t)}{t-u} + (stu \to tus) + (stu \to ust) ,$$

$$G_2(s, t) = \left[\frac{F_1(s, t)}{t-u} - \frac{F_1(t, s)}{s-u} \right] \frac{1}{s-t} + (stu \to tus) + (stu \to ust) \tag{4 1}$$

We shall write $\boldsymbol{G}(s, t)$ for the three-component vector consisting of G_0, G_1 and G_2.

We express $\boldsymbol{G}$ in terms of the symmetric Wanders variables [9]

$$x = -\tfrac{1}{16}(st + tu + us) , \tag{4.2}$$

$$y = \tfrac{1}{64} stu , \tag{4 3}$$

and we write a dispersion relation for $\boldsymbol{G}(x, y)$ on the straight line $y = a(x - x_0)$. As shown in ref. [8], one may re-express this dispersion relation in terms of the variables s, t and u as follows

$$\boldsymbol{G}(s, t) = \boldsymbol{G}(s_1, t_1) + \frac{1}{\pi} \int_4^{\infty} \mathrm{d}s' \, \Delta \boldsymbol{G}(s', t') (s' - t') (2s' + t' - 4)$$

$$\times \left[\frac{1}{(s'-s)(s'-t)(s'-u)} - \frac{1}{(s'-s_1)(s'-t_1)(s'-u_1)} \right] , \tag{4.4}$$

where (s, t) and (s_1, t_1) are two points that map onto the line $y = a(x - x_0)$, which means that

$$\frac{stu}{st + tu + us + 16x_0} = \frac{s_1 t_1 u_1}{s_1 t_1 + t_1 u_1 + u_1 s_1 + 16x_0} = -4a \tag{4.5}$$

In (4.4), (s', t') naturally maps also onto the aforementioned line, from which one deduces that

$$t'(s', a, x_0) = \tfrac{1}{2}\left\{4 - s' + \left[(s'-4)^2 - 16a\,\frac{s'(s'-4) - 16x_0}{s' + 4a}\right]^{\frac{1}{2}}\right\}, \tag{4.6}$$

and $\Delta G(s', t')$ is the discontinuity of $G(s', t')$ across the cut $4 \leqslant s' < \infty$, divided by $2i$

Eq (4.4) is the basic equation given in ref. [8]. It is possible, by going through some algebraic torture, to express (4.4) in the following form, which is very reminiscent of the Cini–Fubini representation [10]:

$$G(s, t) = G(s_1, t_1) + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\,\Delta G(s', t')$$

$$\times \left\{\frac{s^2}{s'-s} + \frac{t^2}{s'-t} + \frac{u^2}{s'-u} - \frac{s_1^2}{s'-s_1} - \frac{t_1^2}{s'-t_1} - \frac{u_1^2}{s'-u_1}\right\} \tag{4.7}$$

One sees that G is manifestly crossing symmetric Also, since

$$t'(s, a, x_0) = t \text{ or } u\,, \tag{4.8}$$

depending on which sign of the square root in (4.6) we choose, it follows that the discontinuities on all three cuts $4 \leqslant s < \infty$, $4 \leqslant t < \infty$, $4 \leqslant u < \infty$, are reproduced correctly by (4.7) The ambiguity (4.8) does not occur in the definition of $\Delta G(s', t')$, because if $F(s, t)$ is Bose symmetric, then

$$\Delta G(s', t') = \Delta G(s', 4 - s' - t')\,, \tag{4.9}$$

and so $\Delta G(s', t')$ is even as a function of

$$z' = 1 + \frac{2t'}{s'-4} = \left\{1 - 16a\,\frac{s'(s'-4) - 16x_0}{(s'-4)^2(s'+4a)}\right\}^{\frac{1}{2}}, \tag{4.10}$$

at fixed s'. Hence the surd disappears

It is convenient to choose the subtraction point

$$s_1 = 2[1 + (1+4x_0)^{\frac{1}{2}}] \stackrel{\text{def}}{=} s_0, \qquad t_1 = 0\,, \tag{4.11}$$

so that (4.7) becomes

$$G(s, t) = G(s_0, 0) + \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'^2} \Delta G(s', t')$$

$$\times \left\{ \frac{s^2}{s' - s} + \frac{t^2}{s' - t} + \frac{u^2}{s' - u} - \frac{s_0^2}{s' - s_0} - \frac{(4 - s_0)^2}{s' - 4 + s_0} \right\} . \tag{4.12}$$

We have lost no generality here, in the sense that (4.7) may be trivially recovered from (4.12) Hence we shall be content to work with the latter equation.

We may remove $G(s_0, 0)$ in favour of $G(4, 0)$, which is related to the scattering lengths, by rewriting (4 12) for the special case $a = 0$, $s = 4$, $t = 0 = t'$ The result is

$$G(s_0, 0) = G(4, 0) - \frac{1}{\pi} \int_4^\infty \frac{ds'}{s'^2} \Delta G(s', 0) \left\{ \frac{16}{s' - 4} - \frac{s_0^2}{s' - s_0} - \frac{(4 - s_0)^2}{s' - 4 + s_0} \right\} , \tag{4 13}$$

which may then be substituted into (4.12). We have thus an equation for $G(s, t)$, in terms of the subtraction constants, $G(4, 0)$, and the discontinuities, $\Delta G(s', t')$, $\Delta G(s', 0)$, and depending upon a parameter s_0. It is important to notice that if ΔG satisfies the Bose symmetry (4.9), then (4.12) defines G as a function with full *stu* crossing symmetry. However, it must be stressed that the representation (4.12) is not valid for all values of s, t and u [8].

It appears at first sight as if there are three subtraction constants, $G_0(4, 0)$, $G_1(4, 0)$ and $G_2(4, 0)$, but in fact G_2 satisfies an unsubtracted relation, so that

$$G_2(4, 0) = \frac{1}{\pi} \int_4^\infty ds' \Delta G_2(s', 0) \left[\frac{1}{s' - 4} + \frac{1}{s'} \right] \tag{4 14}$$

One has therefore just two independent subtraction constants, as in sect. 2 and these may be related to the S-wave scattering lengths as follows

$$a_0 = \tfrac{5}{3} G_0(4, 0) + \tfrac{16}{9} G_1(4, 0) - \tfrac{64}{27} G_2(4, 0) ,$$

$$a_2 = \tfrac{2}{3} G_0(4, 0) - \tfrac{8}{9} G_1(4, 0) + \tfrac{32}{27} G_2(4, 0) \tag{4 15}$$

It is to be remarked that the S- and P-wave contributions to the system (4 12), (4.13), (4.14) and (4 15), just give back the Cini–Fubini approximation (2.28) to the "Cini–Fubini representation" (4.12).

We may set up a mapping like those of sect. 2, in which we first define A in terms of the partial-wave absorptive parts, eq. (2 5). Then ΔG is defined by means of the imaginary parts of eqs (4.1), which we may re-express, by using crossing symmetry, in the form [8]

$$\Delta G_0(s,t) = \tfrac{1}{3}[A_0(s,t) + 2A_2(s,t)] \,,$$

$$\Delta G_1(s,t) = \frac{3s-4}{6(s-t)(s-u)}[2A_0(s,t) - 5A_2(s,t)] + \left[\frac{1}{t-u} - \frac{t-u}{2(s-t)(s-u)}\right]A_1(s,t)$$

$$\Delta G_2(s,t) = -\frac{1}{2(s-t)(s-u)}[2A_0(s,t) - 5A_2(s,t)] + \frac{3(3s-4)}{2(t-u)(s-t)(s-u)}A_1(s,t) \,. \tag{4 16}$$

It is important to notice that if $\mathrm{Im}\, f_l$ vanishes for $I + l$ odd, then $\boldsymbol{A}$ satisfies Bose symmetry, i.e.

$$A_I(s,t) = (-1)^I A_I(s,u) \,, \tag{4 17}$$

and so $\Delta\boldsymbol{G}(s,t)$, defined by (4 16), is automatically symmetrical under the interchange $t \leftrightarrow u$. Hence $\boldsymbol{G}$ may be defined by eqs. (4 12) and (4 13), without any ambiguity, and it will be fully stu symmetrical, as we have seen.

We can calculate $\boldsymbol{F}$ from $\boldsymbol{G}$ by the formulae

$$F_0(s,t) = \tfrac{5}{3}G_0(s,t) + \tfrac{2}{9}(3s-4)G_1(s,t) - \tfrac{2}{27}(3s^2 + 6tu - 16)G_2(s,t) \,,$$

$$F_1(s,t) = \tfrac{1}{9}(t-u)[3G_1(s,t) + (3s-4)G_2(s,t)] \,,$$

$$F_2(s,t) = \tfrac{2}{3}G_0(s,t) - \tfrac{1}{9}(3s-4)G_1(s,t) + \tfrac{1}{27}(3s^2 + 6tu - 16)G_2(s,t) \,. \tag{4 18}$$

The fact that $\boldsymbol{G}$ is fully symmetrical implies that $\boldsymbol{F}$ has the correct stu crossing symmetry. We may project out partial waves, using the half-interval, because Bose symmetry is automatic. Let us summarize the above equations as the mapping

$$\boldsymbol{f}_l(s) = \delta[\mathrm{Im}\, \boldsymbol{f}, l, s] \overset{\mathrm{def}}{=} \tfrac{1}{2}[1 + (-1)^l C_{tu}] \int_0^1 \mathrm{d}z_s P_l(z_s) \boldsymbol{F}(s, t(z_s, s)) \tag{4.19}$$

It is shown in ref. [8] that this representation is valid for any physical s up to 90.20, if we take the parameter $x_0 = 0$. This is therefore a considerable improvement on the 60 of the α-mapping in sect. 2 Indeed, since we could not actually use the α-mapping, in view of the threshold problem, we should really compare 90 20 with 32, the maximum s-value that can be used with the γ-mapping. It is possible to extend the validity of the δ-mapping of (4.19) even further by using also $x_0 = 50.41$. In this case the equation does not hold for s below 39.78, but it can be used up to 125.31. It should be remarked at this point that we cannot regard a and x_0 as independent parameters For a given x_0 and s, we choose to regard a as a function of t, through eq. (4.5). In this way we can cover the required integration interval in (4.19).

It is important to check that the threshold behaviour of $\boldsymbol{f}_l(s)$, as defined by (4 19), is correct. We may immediately replace (4 19) by

$$f_l(s) = \tfrac{1}{2} \int_{-1}^{1} \mathrm{d}z_s P_l(z_s) \boldsymbol{F}(s, t(z_s, s)) , \tag{4 20}$$

because $t - u$ crossing is automatic. We substitute the expressions (4 18) into the above, and use the MRW equation in the form (4 12). We may now expand $\Delta \boldsymbol{G}(s', t')$ in the partial wave series

$$\Delta \boldsymbol{G}(s', t') = \sum_{l'=0}^{\infty} (2l'+1) \Delta \boldsymbol{G}_{l'}(s') P_{l'}(z') , \tag{4 21}$$

with only even values of l', and where z' was defined in (4.10). We separate the Cauchy denominators $s' - t$ and $s' - u$ from the powers of t and u, so that we have to consider the trivial polynomial terms that contribute only to the S and P waves, and then the more complicated terms,

$$\int_{-1}^{1} \mathrm{d}z_s P_l(z_s) P_{l'}(z') , \tag{4 22a}$$

$$\frac{2}{s-4} \int_{-1}^{1} \mathrm{d}z_s P_l(z_s) P_{l'}(z') (z' \pm z_s)^{-1} . \tag{4.22b}$$

We write $P_{l'}(z')$ explicitly as a polynomial in z', involving only even powers, and note that

$$z'^2 = 1 + \frac{4[s'(s'-4) - 16x_0]\, s(s-4)^2(1-z_s^2)}{(s'-4)^2\{(s'-s)(1-z_s^2)(s-4)^2 - 4s'[s(s-4) - 16x_0]\}} \tag{4 23}$$

It is not difficult to see, by means of an expansion in powers of $(s-4)(1 \pm z_s)$, that the expressions (4 22a) and (4.22b) give precisely the threshold behaviour $(s-4)^l$ (since z_s^n is orthogonal to $P_l(z_s)$ on the whole interval, if $n < l$).

We can with advantage use the δ-mapping, combined with unitarity, to define a mapping $\mathrm{Im}\, \boldsymbol{f}_l \to \mathrm{Im}\, \boldsymbol{f}_l'$. In this case, it will be necessary to supply a model for $\mathrm{Im}\, \boldsymbol{f}_l(s)$, $s \geqslant 125$ 31, and some model for the elasticity, $\eta_l(s)$, $16 \leqslant s \leqslant 125$ 31, as well as values for the scattering lengths One no longer has a crossing-symmetry subsidiary constraint upon the input quantities, since $\boldsymbol{F}(s, t)$ is fully crossing symmetric, but in general one has to exclude unwanted singularities that arise from the denominator $s' + 4a$ in (4 10) A necessary constraint is that the fixed point should be independent of x_0. Hence one could think of using the mapping at many different values of x_0 between 0 and 50 41, and one could then vary the input quantities $\mathrm{Im}\, f_l(s)$ (for large s), and $\eta_l(s)$, and also the subtraction constants, in such a way as to minimize the x_0 dependence of the fixed point. In a subsequent paper we propose to investigate the resulting subsidiary conditions in detail.

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After this work was completed, the paper of Auberson and Khuri [12] was brought to our notice. In the "note added in proof" at the end of that paper, it was shown that the new equations reduce to a Cini–Fubini form if the amplitude is completely symmetric. It may be shown in fact that this form is precisely equivalent to our Cini–Fubini representation, for the special case $x_0 = -(a+\frac{1}{9})/3a$. Hence the Mahoux–Roy–Wanders equation (for the special case $x_0 = -(a+\frac{1}{9})/3a$) is the same as the Auberson–Khuri equation (for the special case that the amplitude is *stu* symmetric). We thank Dr J.S Frederiksen for very helpful discussions of this point.

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